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On a Conjecture of Z. Ditzian

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A conjecture of Z. Ditzian on Bernstein polynomials is proved. This yields additional information on the problem of characterizing the rate of convergence for Bernstein polynomials.

The Bernstein polynomials on $C[0, 1]$ are given by

$$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\, P_{nk}(x), \tag{1}$$

where

$$P_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}. \tag{2}$$

The relation between the rate of the polynomials' convergence and the smoothness of the function they approximate has been investigated in many papers. Some of these results can be stated in the following theorem.

THEOREM A. *For $f \in C[0, 1]$, $X = x(1-x)$, $0 \leqslant \alpha < 2$, $0 < \beta < 2$, the following statements are equivalent*:

$$(1)\quad X^{-\alpha/2} |B_n(f, x) - f(x)| \leqslant M_1 n^{-\beta/2}; \tag{3}$$

$$(2)\quad X^{-\alpha/2} |f(x-t) - 2f(x) + f(x+t)| \leqslant M_2 (t^2/X)^{\beta/2}. \tag{4}$$

Here M_1, M_2 are constants independent of n, x, and t.

In 1972, H. Berens and G. G. Lorentz [1] proved this theorem for $\alpha = \beta$. M. Becker [2] gave an elementary proof of this case. The case $\alpha = 0$ was

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proved by Z. Ditzian [3] and V. Totik [9]. Z. Ditzian [4] proved this theorem for $0 \leqslant \alpha \leqslant \beta < 2$ as well in 1980.

Theorem A had remained open for the case $2 > \alpha > \beta > 0$ for a few years. In 1987, Z. Ditzian [5] gave the proof of this case with the restriction $\alpha + \beta \leqslant 2$. In that paper he also gave the following conjecture which allows us to complete the proof for the remaining cases. In this paper we give a proof of this conjecture, and show that Theorem A holds.

THEOREM (Conjecture of Z. Ditzian). *Suppose for* $0 < \beta < \alpha < 2$, $f \in C[0, 1]$ *satisfies* $f(0) = f(1) = 0$ *and*

$$X^{-\alpha/2} |f(x-t) - 2f(x) + f(x+t)| \leqslant (t^2/X)^{\beta/2} \tag{5}$$

for any $x \in (0, 1)$, $x \pm t \in [0, 1]$; *then we must have for the case* $\alpha + \beta > 2$

(1) $f(x) = A_1 x + f_1(x)$,

where $f_1(x) = O(x^{(\alpha+\beta)/2})$ *as* $x \to 0^+$;

(2) $f(x) = A_2(1-x) + f_2(x)$,

where $f_2(x) = O((1-x)^{(\alpha+\beta)/2})$ *as* $x \to 1^-$.

Proof. We need only prove (1).

From (5), we have for $x \in (0, \frac{1}{2})$

$$|f(x) - 2f(x/2)| \leqslant x^{(\alpha+\beta)/2},$$

which implies

$$|f(x)/x - 2f(x/2)/x| \leqslant x^{(\alpha+\beta)/2 - 1}. \tag{6}$$

Define $g_n(x) = 2^n f(x/2^n)/x$ for $x \in [\frac{1}{4}, \frac{1}{2}]$, then $g_n \in C[\frac{1}{4}, \frac{1}{2}]$. For these functions we have from (6) for $n \in N$, $p \in N$, $x \in [\frac{1}{4}, \frac{1}{2}]$

$$\begin{aligned} |g_n(x) - g_{n+p}(x)| &= \left| \sum_{k=n+1}^{n+p} (2^k f(x/2^k)/x - 2^{k-1} f(x/2^{k-1})/x) \right| \\ &\leqslant \sum_{k=n+1}^{n+p} (x/2^{k-1})^{(\alpha+\beta)/2 - 1} \\ &\leqslant \sum_{k=n+1}^{n+p} (2^{1-(\alpha+\beta)/2})^{k-1}. \end{aligned}$$

Thus by $\alpha + \beta > 2$, we know that $\{g_n\}$ is convergent uniformly on $[\frac{1}{4}, \frac{1}{2}]$. So we can define

$$g(x) = \lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} 2^n f(x/2^n)/x, \tag{7}$$

then $g \in C[\frac{1}{4}, \frac{1}{2}]$.

Now we prove that g is constant on $[\frac{1}{4}, \frac{1}{2}]$.

Let $h(x) = xg(x)$. From (5), for $x \in [\frac{1}{4}, \frac{1}{2}]$, $t \in (0, \frac{1}{8})$, $x \pm t \in [\frac{1}{4}, \frac{1}{2}]$, and $n \in N$, we have

$$|f((x-t)/2^n) - 2f(x/2^n) + f((x+t)/2^n)| \leqslant t^\beta x^{(\alpha-\beta)/2} 2^{-n(\alpha+\beta)/2}$$

and

$$|(x-t)\, g_n(x-t) - 2xg_n(x) + (x+t)\, g_n(x+t)| \leqslant t^\beta x^{(\alpha-\beta)/2} 2^{n(1-(\alpha+\beta)/2)}.$$

By letting $n \to \infty$, we have from the definition of h

$$h(x-t) - 2h(x) + h(x+t) = 0,$$

where x, $x \pm t \in [\frac{1}{4}, \frac{1}{2}]$ are arbitrary. From this fact, we know [8] that h is linear on $[\frac{1}{4}, \frac{1}{2}]$, say,

$$h(x) = ax + b.$$

Hence $g(x) = a + b/x$.

Now from (7), we know that

$$g(\tfrac{1}{2}) = g(\tfrac{1}{4}) = \lim_{n \to \infty} 2^n f(2^{-n});$$

therefore, we must have $b = 0$ and

$$g(x) = a. \tag{8}$$

On the other hand, as for every $x \in (0, \frac{1}{4})$, there exists n_0 such that $x2^{n_0} \in [\frac{1}{4}, \frac{1}{2}]$; in view of (7) and (8) we get

$$\lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)\Big/x = \lim_{n - n_0 \to \infty} 2^{n-n_0} f\left(\frac{x2^{n_0}}{2^{n-n_0}}\right)\Big/(x2^{n_0}) = a,\ x \in (0, \tfrac{1}{4}).$$

Thus, by (6),

$$\left| f(x)/x - 2^n f\left(\frac{x}{2^n}\right)\Big/x \right|$$
$$\leqslant \sum_{k=1}^{n} \left| 2^k f\left(\frac{x}{2^k}\right)\Big/x - 2^{k-1} f\left(\frac{x}{2^{k-1}}\right)\Big/x \right| \leqslant Cx^{(\alpha+\beta)/2-1},$$

where the constant C is independent of n and x by $\alpha + \beta > 2$.

By letting $n \to \infty$, we have for any $x \in (0, \frac{1}{4})$

$$|f(x) - ax| \leqslant Cx^{(\alpha+\beta)/2}.$$

The proof is complete.

Proof of Theorem A (For $2 > \alpha > \beta > 0$ and $\alpha + \beta > 2$). By [1], we need only prove that (4) implies (3).

It is sufficient for us to prove the theorem on $[0, \frac{1}{2}]$, and to prove from (4) that

$$|B_n(f, x) - f(x)| \leqslant M_1 x^{\alpha/2} n^{-\beta/2} \tag{9}$$

with a constant M_1 independent of n and $x \in [0, \frac{1}{2}]$.

In this case we can assume that $f|_{[3/4, 1]} = 0$ and $|f(x)| \leqslant Mx^{(\alpha+\beta)/2}$ with the constant M independent of $x \in [0, 1]$. We divide the proof into two parts.

For $0 < x \leqslant 2/n$, the proof is easy:

$$\begin{aligned} |B_n(f, x) - f(x)| &\leqslant M \sum_{k=0}^{n} (k/n)^{(\alpha+\beta)/2-1} kP_{nk}(x)/n + Mx^{(\alpha+\beta)/2} \\ &\leqslant Mx \sum_{k=1}^{n} (k/n)^{(\alpha+\beta)/2-1} P_{n-1,k-1}(x) + Mx^{(\alpha+\beta)/2} \\ &\leqslant Mxn^{1-(\alpha+\beta)/2} + Mx\left(\sum_{k=2}^{n} \frac{2(k-1)}{n-1} P_{n-1,k-1}(x)\right)^{(\alpha+\beta)/2-1} \\ &\quad + Mx^{(\alpha+\beta)/2} \\ &\leqslant 8Mx^{\alpha/2} n^{-\beta/2}. \end{aligned} \tag{10}$$

For $\frac{1}{2} \geqslant x > 2/n$, $n \geqslant 8$, let $d = (x/n)^{1/2} < x$ and define a Steklov function as

$$f_d(t) = (2/d)^2 \iint_0^{d/2} (2f(t+u+v) - f(t+2u+2v))\, du\, dv, \tag{11}$$

where f has been extended to $y \in [1, 2]$ as $f(y) = 0$.

Then we have

$$\begin{aligned} |f(t) - f_d(t)| &\leqslant M_2 (2/d)^2 \iint_0^{d/2} (u+v)^\beta\, (t+u+v)^{(\alpha-\beta)/2}\, du\, dv \\ &\leqslant M_2 d^\beta (t+d)^{(\alpha-\beta)/2} \end{aligned}$$

and

$$|f_d''(t)| \leqslant 9M_2 d^{\beta-2} (t+d)^{(\alpha-\beta)/2} \leqslant 9M_2 d^{\beta-2} (d^{(\alpha-\beta)/2} + t^{(\alpha-\beta)/2}). \tag{12}$$

Thus we can estimate (9) as

$$
\begin{aligned}
|B_n(f,x)-f(x)| &\leqslant |f_d(x)-f(x)| + |B_n(f_d-f,x)| + |B_n(f_d,x)-f_d(x)| \\
&\leqslant 2M_2 d^{\beta} x^{(\alpha-\beta)/2} + M_2 d^{\beta} B_n((t+d)^{(\alpha-\beta)/2}, x) \\
&\quad + 9M_2 d^{\beta-2} B_n\left(\int_x^t (t-u)(d^{(\alpha-\beta)/2} + u^{(\alpha-\beta)/2})\, du, x\right) \\
&\leqslant 4M_2 x^{\alpha/2} n^{-\beta/2} + 9M_2 d^{\beta-2}\Bigg(d^{(\alpha-\beta)/2} x/n \\
&\quad + (x/n)^{1-(\alpha-\beta)/2}\left(B_n\left(\int_x^t (t-u) u\, du, x\right)\right)^{(\alpha-\beta)/2}\Bigg) \\
&\leqslant 50 M_2 x^{\alpha/2} n^{-\beta/2},
\end{aligned}
\tag{13}
$$

where we have used the estimate

$$
\begin{aligned}
B_n\left(\int_x^t (t-u)u\, du, x\right) &= B_n((t^3 - tx^2)/2 - (t^3 - x^3)/3, x) \\
&= (B_n(t^3, x) - x^3)/6 \\
&\leqslant x n^{-2} + 3x^2 n^{-1} \\
&\leqslant 4x^2/n.
\end{aligned}
$$

Combining (10) and (13), we have proved (9) for $n \geqslant 8$.

For $n < 8$, it is trivial since $|B_n(f,x) - f(x)| \leqslant B_n(Mt, x) + Mx \leqslant 2Mx \leqslant 14Mx^{\alpha/2} n^{-\beta/2}$. Our proof is therefore complete.

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